EMBEDDING THEOREMS FOR TWO-PARAMETER GROUPS OF FORMAL POWER SERIES AND RELATED PROBLEMS

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ABSTRACT

The classical problem of analytic iteration is that of embedding analytic functions in one-parameter Lie groups of formal power series. The main purpose of the present paper is to consider similar problems for two-parameter groups.

These problems are closely related to problems concerning conjugate power series and conjugate one parameter groups of such series, to which the first sections of the paper are devoted.

As an application of the conjugacy theorems and the embedding theorems we bring an algebraic characterization of the class of the two-parameter groups.

1. Introduction and definitions

1.1. Let Σ^F denote the linear algebra of formal power series (over the field of complex numbers) having the form

$$F(z) = \sum_{q=0}^{\infty} f_q z^q. \tag{1}$$

(Operations on formal power series are defined in [6], ch.1). Σ^F is equipped with the metric

$$\rho(F, G) = \sum_{q=0}^{\infty} 2^{-q} \frac{|f_q - g_q|}{1 + |f_q - g_q|} \tag{2}$$

for $F, G \in \Sigma^F$. By Σ_0^F we denote the subalgebra of Σ^F which contains the series of the form (1) with $f_0 = f_1 = 0$. By Σ we denote the subalgebra of Σ^F which contains the power series of the form (1) having a non-zero radius of convergence. We further define $\Sigma_0 = \Sigma \cap \Sigma_0^F$. By $Q[\Sigma^F]$ we shall denote the quotients field of Σ^F.

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Let:

$$F(z,s) = \sum_{q=0}^{\infty} f_q(s)z^q \tag{3}$$

be a family of formal power series depending on a complex parameter s, $s \in D$, D being an open set of the complex plane. We say, that $F(z,s)$ is analytically dependent on s, if the limit

$$\frac{\partial F(z,s)}{\partial s} = \lim_{h \to 0} \frac{F(z,s+h) - F(z,s)}{h} = \sum_{q=0}^{\infty} f'_q(s)z^q$$

exists in the metric of Σ^F for every $s \in D$. This limit exists if and only if all the coefficients $f_q(s)$, $q = 0, 1, 2, \cdots$ are analytic functions of s in D.

1.2. Let Ω^F denote the subset of Σ^F, which contains the power series of the form

$$F(z) = \sum_{q=1}^{\infty} f_q z^q, \qquad f_1 \neq 0. \tag{4}$$

Ω^F forms a topological group with respect to formal substitution of formal power series, and the topology induced in Ω^F from Σ^F.

By Ω we denote the subgroup of Ω^F constituted by the power series of Ω^F with a non-zero radius of convergence. By Ω_1^F we denote the subgroup of Ω^F which contains the power series of the form (4) with $f_1 = 1$. We also define Ω_1 by $\Omega_1 = \Omega \cap \Omega_1^F$.

By $\Omega^F(n)$ we denote the subgroup of Ω^F containing all the series of the form

$$F(z) = f_1 z + \sum_{q=n+1}^{\infty} f_q z^q \tag{5}$$

we further define $\Omega_1^F(n) = \Omega_1^F \cap \Omega^F(n)$.

We note, that the group Ω^F, though not being locally compact, can be approximated by finite dimensional Lie groups; that is, every neighborhood of the identity element of Ω^F contains a normal subgroup G such that the quotient group Ω^F/G is a finite dimensional Lie group. Indeed, for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for $n > N(\varepsilon)$, $\Omega_1^F(n)$ is contained in the open ball with center at the identity element and radius ε, and for all n, $\Omega^F/\Omega_1^F(n)$ is a finite dimensional Lie group. (For approximation of locally compact groups by Lie groups see [12].)

1.3. A subgroup of Ω^F is said to be an *analytic one parameter subgroup of* Ω^F (a O.P. subgroup) if its elements can be written as a one-parameter family of $\Omega^F: F(z,s)$, where s ranges in some domain D in the complex plane so that

1) $F(z,s)$ is analytically dependent on s.

2) There is a function $\phi(s,t)$, analytic in both s and t for $s,t\in D$, which is called the "multiplication table", such that

$$F[F(z,s),t] = F[z,\phi(s,t)] \tag{6}$$

holds for every s, $t\in D$.
We shall use two "multiplication tables": the *canonical* multiplication table, $\phi(s,t) = s+t$ and the *standard* multiplication table, $\phi(a,b) = a+b+ab$. By $\tilde{F}(z,s)$ we shall denote a O.P. subgroup with canonical parametrization, and by $\hat{F}(z,a)$ a O.P. subgroup with standard parametrization.

1.4. E. Jabotinsky [10] proved, that for every $F(z)\in\Omega_1^F$ there exists a unique O.P. subgroup of Ω^F which contains it; more specifically, there exists a unique O.P. subgroup of Ω^F, $\tilde{F}(z,s)$ (with canonical parameter), such that

$$\tilde{F}(z,1) = F(z). \tag{7}$$

Moreover, the O.P. subgroup $\tilde{F}(z,s)$ is contained in Ω_1^F.

Using this result, Jabotinsky defined a mapping $J:\Omega_1^F\to\Sigma_0^F$ by:

$$J[F(z)] = \left(\frac{\partial\tilde{F}(z,s)}{\partial s}\right)_{s=0} = \sum_{q=1}^{\infty} f_q'(0)z^q = \sum_{q=1}^{\infty} l_{q-1}z^q \tag{8}$$

where $\tilde{F}(z,s)$ is the unique O.P. subgroup satisfying (7). The mapping J is proved to be a one-to-one correspondence between Ω_1^F and Σ_0^F [10].

The coefficients $\{l_q\}_0^\infty$ in (8) form the l-sequence of the series $F(z)$. Jabotinsky [10] gives the following expression for the terms of the l-sequence

$$l_k = \sum_{j=1}^{k}\frac{(-1)^j}{j!}C_{k,j}f_{k+1}^{(j)} \tag{9}$$

where $f_q^{(j)}$ are the coefficients of the jth formal iterate of $F(z)$:

$$F^{[j]}(z) = \sum_{q=1}^{\infty} f_q^{(j)}z^q.$$

In the case of $F(z)\in\Omega_1$, that is $F(z)$ represents an analytic function, it was shown ([2], [7]), that only two cases are possible:

A) The O.P. subgroup of Ω_1^F satisfying (7) is contained in Ω_1; $\tilde{F}(z,s)$ is then an analytic function of s. In this case we say that the series $F(z)$ is *embeddable* (in a O.P. subgroup of Ω_1).

B) The O.P. subgroup of Ω_1^F satisfying (7) meets Ω_1 only for a set of parameters having the two dimensional measure 0 in the complex plane and the one dimensional measure 0 on the real axis in the parameter plane. In this case we say that the series $F(z)$ is *non-embeddable.*

Thus Ω_1 appears as the union of two disjoint classes: the class Ω_1^A of all the embeddable elements of Ω_1, and the class Ω_1^B of the non-embeddable ones. Both classes are not empty (see, e.g. [1], [11], [14]). It was proved ([3], [4], [17]) that all the series of Ω_1, representing entire or meromorphic functions, except for the Moebius linear functions (as well as a wide class of algebraic functions) belong to the class Ω_1^B.

Erdös and Jabotinsky proved that $F(z) \in \Omega_1^A$ iff $J[F(z)] \in \Sigma_0$ [7]. It is known, that Ω_1^A and Ω_1^B are not subgroups of Ω_1 [14].

We define also the class Ω^A to be the subset of Ω containing those elements which belong to some O.P. subgroup of Ω; we also define $\Omega^B = \Omega - \Omega^A$.

1.5. Let $F(z)$, $G(z) \in \Omega^F$, and Γ be a subgroup of Ω^F. We say, that $F(z)$ and $G(z)$ are *conjugate with respect to* Γ, if there exists $\phi(z) \in \Gamma$ such that

$$\phi[F(z)] = G[\phi(z)]. \tag{10}$$

By comparing coefficients in (10), we see that if $F(z)$ is conjugate to $G(z) \in \Omega_1^F$ with respect to Ω^F, then $F(z) \in \Omega_1^F$; that is, Ω_1^F is self conjugate with respect to Ω^F.

If $F(z,s)$ is a O.P. subgroup of Ω^F, and $\phi(z) \in \Omega^F$, then $G(z,s) = \phi^{-1}\{F[\phi(z),s]\}$ will also be a O.P. subgroup of Ω^F; and if, moreover, $F(z,s)$ is a subgroup of Ω and $\phi(z) \in \Omega$, $G(z,s)$ will be a subgroup of Ω. If follows, that Ω_1^A and Ω_1^B are self conjugate with respect to Ω.

1.6. We define the *boundary* of $\Omega_1^F(n)$ by

$\partial\Omega_1^F(n) = \Omega_1^F(n) - \Omega_1^F(n+1)$. $F(z) \in \Omega_1^F$ is said to be of *type* n iff $F(z) \in \partial\Omega_1^F(n)$. $F(z) \in \Omega_1^F$ is thus of type n iff it has the form

$$F(z) = z + \sum_{q=n+1}^{\infty} f_q z^q, \qquad f_{n+1} \neq 0.$$

If $F(z) \in \Omega_1^F$ is of type n we shall write $\tau(F) = n$. If $J[F(z)] = \sum_{q=1}^{\infty} l_{q-1} z^q$, it follows from (9) that $\tau(F) = n$ iff

(11) $$l_0 = l_1 = \cdots = l_{n-1} = 0, \; l_n \neq 0.$$

1.7. Let $F(z) \in \Omega_1^F$ and $L(z) = J[F(z)]$. We define the *characteristic number* of $F(z)$ by

(12) $$\rho(F) = \operatorname*{Res}_{z=0} \frac{1}{L(z)}$$

where the residue is understood to be the formal residue in $Q[\Sigma^F]$. We put $\rho(F) = \infty$ for $F(z) = z$.

We note, that if $\tau(F) = p$, $\rho(F)$ depends upon $l_0, l_1, \cdots, l_{2p}$, where $\{l_q\}_0^\infty$ is the l-sequence of $F(z)$.

2. Conjugate series of Ω_1^F

2.1 It is our purpose to prove

THEOREM 1. *Let $F(z)$, $G(z) \in \Omega_1^F$. $F(z)$ and $G(z)$ are conjugate with respect to Ω^F iff both are of the same type and have the same characteristic number.*

In order to prove Theorem 1 we first prove

LEMMA 1. *Let $F(z)$, $G(z) \in \Omega_1^F$, $L(z) = J[F(z)]$, $M(z) = J[G(z)]$, and $\tilde{F}(z,s)$, $\tilde{G}(z,s)$ the unique O.P. subgroups of Ω^F satisfying $\tilde{F}(z,1) = F(z)$, $\tilde{G}(z,1) = G(z)$.*

If $\phi(z) \in \Omega^F$ satisfies any one of the following equations

(13) $$\phi[F(z)] = G[\phi(z)]$$

(14) $$\phi[\tilde{F}(z,s)] = \tilde{G}[\phi(z),s]$$

(15) $$\phi'(z) \,.\, L(z) = M[\phi(z)]$$

it satisfies the other two equations as well.

PROOF. I) Suppose that $\phi(z) \in \Omega^F$ satisfies (13). Define the O.P. subgroup of Ω^F $\tilde{H}(z,s)$ by

(16) $$\tilde{H}(z,s) = \phi^{-1}\{G[\phi(z),s]\}.$$

Putting $s = 1$ we find, using (13), that $\tilde{H}(z,1) = F(z)$ and hence $\tilde{H}(z,s) \equiv \tilde{F}(z,s)$ for all s, and (14) follows from (16).

II) Differentiating (14) with respect to s and putting $s = 0$ we get (15).

III) Suppose that $\phi(z) \in \Omega^F$ satisfies (15). Define $H(z) \in \Omega_1^F$ by

(17) $$H(z) = \phi^{-1}[G\{\phi(z)\}].$$

Denoting $N(z) = J[H(z)]$ we have, by Parts I and II

(18) $$\phi'(z) . N(z) = M[\phi(z)].$$

From (15) and (18) we have $J[H(z)] = J[F(z)]$ and as the mapping $J: \Omega_1^F \to \Sigma_0^F$ is one-to-one, we have: $H(z) = F(z)$ and this and (17) imply (13), completing the proof.

2.2. Proof of Theorem 1.

Denote $L(z) = J[F(z)] = \sum_{q=1}^{\infty} l_{q-1} z^q$, $M(z) = J[G(z)] = \sum_{q=1}^{\infty} m_{q-1} z^q$. In view of Lemma 1 it is sufficient to prove that a necessary and suffieient condition for the existence of $\phi(z) \in \Omega^F$ such that (15) is satisfied is $\tau(F) = \tau(G)$ and $\rho(F) = \rho(G)$.

I) We shall first show, that a necessary condition for the existence of $\phi(z) = \sum_{q=1}^{\infty} \phi_q z^q \in \Omega^F$ such that (15) is satisfied is that $\tau(F) = \tau(G)$. Suppose that $\tau(F) = p$ and $\tau(G) = r$. The first non-zero term on the left side of (15) is $\phi_1 l_p z^{p+1}$, while the first non-zero term on the right is $m_r \phi_1^{r+1} z^{r+1}$, hence $p = r$.

We note, that comparing the coefficients of these terms we get

$$m_p \phi_1^p = l_p. \tag{19}$$

II) We shall show, that a necessary condition for $F(z) \in \partial\Omega_1^F(p)$ to be conjugate to $H_{p,\lambda}(z) \in \Omega_1^F$ defined by

$$\{J[H_{p,\lambda}(z)]\}^{-1} = -\frac{p}{z^{p+1}} + \frac{\lambda}{z} \tag{20}$$

is that $\rho(F) = \lambda$ (it is clear that $\tau[H_{p,\lambda}(z)] = p$ and $\rho[H_{p,\lambda}(z)] = \lambda$).

Equation (15) with $G(z) = H_{p,\lambda}(z)$, that is, $M(z) = (-pz^{-p-1} + \lambda z^{-1})^{-1}$ takes the form

$$-\frac{p\phi'(z)}{[\phi(z)]^{p+1}} + \frac{\lambda\phi'(z)}{\phi(z)} = \frac{1}{L(z)} \tag{21}$$

comparing the formal residues on both sides at $z = 0$ and using the definition (12) we get $\lambda = \rho(F)$, and hence the condition is necessary.

III) We shall prove now that if $F(z) \in \partial\Omega_1^F(p)$ and $\rho(F) = \lambda$ then it is conjugate with respect to Ω^F to $H_{p,\lambda}(z)$ defined by (20). Because of the transitivity of the conjugacy relation, the theorem will follow.

It is sufficient to show that (21) admits a solution $\phi(z) = \sum_{q=1}^{\infty} \phi_q z^q \in \Omega^F$. After a formal integration of (21) in $Q[\Sigma^F]$ we can write the formal equation

$$\left(\frac{\phi}{z}\right)^{-p} + \lambda z^p Ln \frac{\phi}{z} = \Gamma(z) \tag{22}$$

where $\Gamma(z) \in \Sigma^F$. Denoting

(23) $$\psi(z) = \frac{\phi(z)}{z} - \phi_1$$

(22) takes the form

(24) $$[\phi_1 + \psi(z)]^{-p} + \lambda z^p Ln[\phi_1 + \psi(z)] = \Gamma(z).$$

We compare powers of z in both sides of (24). The free terms determine ϕ_1. We note, that the fact that $F(z)$ was of type p implies that the free term of $\Gamma(z)$ is different from zero, and hence $\phi_1 \neq 0$. In order to compare the other coefficients on both sides of (24), we denote: $\psi(z) = \Sigma_{q=1}^{\infty}\psi_q z^q$, and get equations for the coefficients ψ_q of the form

(25) $$-p\psi_q = \Lambda_q(\psi_1, \cdots, \psi_{q-1}).$$

$\psi(z)$ can be constructed by solving the system (25); using (23) we can obtain the required series $\phi(z)$, and the theorem is proved.

2.3. It can be asked, when are two given elements of Ω_1^F conjugate with respect to Ω_1^F; putting $\phi_1 = 1$ in Equation (19) we get

THEOREM 2. *Let* $F(z)$, $G(z) \in \Omega_1^F$, *and* $J[F(z)] = \Sigma_{q=1}^{\infty} l_{q-1} z^q$, $J[G(z)] = \Sigma_{q=1}^{\infty} m_{q-1} z^q$. *$F(z)$ and $G(z)$ are conjugate with respect to Ω_1^F iff both are of the same type p, $\rho(F) = \rho(G)$, and $l_p = m_p$.*

NOTE. It can be asked, when are two elements of $\partial\Omega_1^F(p)$ conjugate with respect to $\Omega^F(p)$. By putting in (15) $\phi(z) = \lambda z + z^{p+1}\psi(z)$, we find that a necessary and sufficient condition for the existence of a solution $\psi(z) \in \Sigma^F$ is

(26) $$l_q = \lambda^q m_q \qquad q = 1, 2, \cdots, 2p$$

where $\lambda \neq 0$ is some complex number.

3. A convergence property of Ω_1^A

3.1. THEOREM 3. *If $F(z)$, $G(z) \in \Omega_1^A$ are conjugate with respect to Ω^F, then $F(z)$ and $G(z)$ are conjugate with respect to Ω; more specifically, if $F(z), G(z) \in \Omega_1^A$ and $\phi[F(z)] = G[\phi(z)]$ with $\phi(z) \in \Omega^F$, then $\phi(z) \in \Omega$. Ω_1^A is a maximal set with respect to this property in Ω_1^F.*

PROOF. (I) In view of Lemma 1 and the fact that $F(z) \in \Omega_1^A$ iff $J[F(z)] \in \Sigma_0$, the property of Ω_1^A will be proved once we show that if $\phi(z) = \Sigma_{q=1}^{\infty} \phi_q z^q \in \Omega^F$ satisfies the equation

(27) $$\phi'(z)L(z) = M[\phi(z)]$$

where $L(z)$, $M(z) \in \Sigma_0$, then $\phi(z) \in \Omega$.

By Theorem 1, if (27) is satisfied then both $L(z)$ and $M(z)$ start with the same power of z, say, $p+1$: $L(z)=\Sigma_{q=p}^{\infty} l_q z^{q+1}$, $M(z)=\Sigma_{q=p}^{\infty} m_q z^{q+1}$ where $l_p, m_p \neq 0$. By (19) we have

$$m_p \phi_1^p = l_p . \tag{28}$$

Define $\theta(z) \in \Sigma^F$ by

$$\phi(z) = \sum_{q=1}^{p} \phi_q z^q + z^{p+1}\theta(z). \tag{29}$$

Introducing (29) into (27) we get

$$\theta' = \frac{M\left[\sum_{q=1}^{p} \phi_q z^q + z^{p+1}\theta\right] - (p+1)z^p\theta L(z) - \left[\sum_{q=1}^{p} q\phi_q z^{q-1}\right] L(z)}{z^{p+1}L(z)} \tag{30}$$

As $\theta' \in \Sigma^F$, the nominator of the right side of (30) should be divisible by z^{2p+2}, because the denominator starts with the term $l_p z^{2p+2}$. We note, that the nominator does not contain terms of the form $\kappa\theta z^r$ with $r < 2p+2$, because the terms $(p+1)m_p\phi_1^p z^{2p+1}\theta$ and $-(p+1)l_p z^{2p+1}\theta$ conceal each other by (28).

We may regard the right side of (30) as an analytic function of the variables z and θ near the origin, and hence the differential equation (30) admits a solution $\theta = \theta(z)$ analytic in the neighborhood of $z=0$; hence from (29) we have $\phi(z) \in \Omega$.

(II) We want to show that Ω_1^A is a maximal set in Ω_1^F with respect to the above property. Take any set S such that: $\Omega_1^A \subset S \subset \Omega_1^F$. Take $F(z) \in S - \Omega_1^A$ and let $J[F(z)] = \Sigma_{q=p}^{\infty} l_{q-1} z^q$, $l_{p-1} \neq 0$. Define $G(z) \in \Omega_1^A$ by $J[G(z)] = \Sigma_{q=p}^{2p} l_{q-1} z^q$. By Theorem 1 $F(z)$ and $G(z)$ are conjugate with respect to Ω^F, but $F(z)$ and $G(z)$ are not conjugate with respect to Ω.

3.2. We note that Theorem 3 becomes false when we replace the class Ω_1^A by Ω^A. Indeed, take $F(z) = G(z) = -z \in \Omega^A$ (we have $\hat{F}(z,-2) = -z$ where $\hat{F}(z,a) = (1+a)z$). We have $\phi(-z) = -\phi(z)$ for any series $\phi(z) = \Sigma_{q=1}^{\infty} \phi_{2q-1} z^{2q-1}$, even if $\phi(z) \notin \Omega$.

3.3. We bring an exanple to show that Ω_1^B does not have the above property. Take

$$F(z) = e^z - 1 = \sum_{q=1}^{\infty} \frac{z^q}{q!}$$

$$G(z) = z + \frac{z^2}{4} + \frac{z^3}{48} + \frac{1}{4}\left(z + \frac{z^2}{4} + \frac{z^3}{48}\right)^2 + \frac{1}{48}\left(z + \frac{z^2}{4} + \frac{z^3}{48}\right)^3$$

$F(z)$ and $G(z)$ belong to Ω_1^B (being expansions of entire functions). We have $f_q = g_q = 1/q!$ for $q = 1,2,3$; hence from (9) follows $l_q = m_q$ for $q = 0,1,2$. As $F(z)$ and $G(z)$ are of type 1, it follows (by Theorem 1) that $F(z)$ and $G(z)$ are conjugate with respect to Ω^F. We want to show that $F(z)$ and $G(z)$ are not conjugate with respect to Ω. Baker [1] showed, that there is no element $T(z) \in \Omega$ such that $T[T(z)] = F(z)$. Suppose that $F(z) = \phi^{-1}\{G[\phi(z)]\}$ with $\phi(z) \in \Omega$. Denote $S(z) = z + (z^2/4) + (z^3/48)$, and $T(z) = \phi^{-1}\ \{S[\phi(z)]\} \in \Omega$. We have then $T[T(z)] = F(z)$, which is a contradiction.

4. Conjugate O.P. subgroups of Ω^F

4.1. There are two classes of O.P. subgroups of Ω^F: the simply connected subgroups (which are subgroups of Ω_1^F) and the non-simply connected subgroups (the intersection of which with Ω_1^F is just the identity element) [16].

Every non-simply connected O.P. subgroup of Ω^F can be uniquely represented in the form

$$\hat{F}(z,a) = \phi^{-1}[(1+a)\cdot\phi(z)] \tag{31}$$

where $\phi(z) \in \Omega_1^F$. $\hat{F}(z,a)$ is a subgroup of Ω iff $\phi(z) \in \Omega$ [16].

From (31) follows, that every two non-simply connected O.P. subgroups of Ω^F are conjugate with respect to Ω^F, that is, for any two such subgroups $\hat{F}(z,a)$ and $\hat{G}(z,a)$ there exists $\psi(z) \in \Omega^F$ such that

$$\psi[\hat{F}(z,a)] = \hat{G}[\psi(z),a]. \tag{32}$$

Moreover, if the two subgroups are contained in Ω, then $\psi(z) \in \Omega$.

We note, that the correspondence between the elements of the subgroup $\hat{F}(z,a)$ and the elements of the subgroup $\hat{G}(z,a)$ given by (32) is an isomorphism, induced by an inner automorphism of Ω^F. From (31) follows, that every automorphism of the subgroup $\hat{F}(z,a)$ induced by an inner automorphism of Ω^F is the identity mapping of $\hat{F}(z,a)$ on itself.

4.2. We turn now to find when are two given O.P. subgroups of Ω_1^F conjugate with respect to Ω^F.

We say, that a O.P. subgroup of Ω_1^F is *of type n*, iff it contains an element of type n.

From the definition of the operator J follows the relation

$$J[\tilde{F}(z,s)] = sJ[\tilde{F}(z,1)] \tag{33}$$

(see [10]). From here follows, that if a O.P. subgroup of Ω_1^F is of type n, all its elements, except for the identity element are of type n; that is, every O.P. subgroup of Ω_1^F of type n is contained in the set $\partial\Omega_1^F(n) \cup \{z\}$.

A O.P. subgroup of Ω_1^F is said to be of the *first kind*, iff it contains an element having the characteristic number 0. As from (33) follows that for $s \neq 0$ we have $\rho[\tilde{F}(z,s)] = s^{-1}\rho[\tilde{F}(z,1)]$, we conclude that in a O.P. subgroup of the first kind all the elements, except for the identity element, have the characteristic number 0.

A O.P. subgroup of Ω_1^F is said to be of the *second kind*, iff it contains at least one element having a finite characteristic number different from 0. From (33) we have, that in a O.P. subgroup of the second kind for any given complex number $\lambda \neq 0$, there is exactly one element $F(z)$ such that $\rho(F) = \lambda$.

We now can state

THEOREM 4. *Two O.P. subgroups of Ω_1^F are conjugate with respect to Ω^F iff both are of the same type and of the same kind; two O.P. subgroups of Ω_1 are conjugate with respect to Ω iff they are conjugate with respect to Ω^F.*

PROOF. A O.P. subgroup of Ω_1^F will contain an element which is of the same type and has the same characteristic number as some element belonging to another O.P. subgroup of Ω_1^F iff the two O.P. subgroups are of the same type and of the same kind; by Theorem 1 these two elements are conjugate with respect to Ω^F. By Lemma 1 this is equivalent to the conjugacy of the two O.P. subgroups.

If, moreover, both subgroups are contained in Ω_1, all their elements belong to Ω_1^A, and hence, by Theorem 3, conjugacy with respect to Ω^F implies conjugacy with respect to Ω, completing the proof.

4.3. The following theorem illustrates two properties of the O.P. subgroups of Ω_1^F of the first kind; it is readily seen that the theorem is false for subgroups of the second kind.

THEOREM 5. I) *Two O.P. subgroups of Ω_1^F of the first kind which are conjugate with respect to Ω^F are conjugate with respect to Ω_1^F.*

II) *For any two elements g_1, g_2 different from the identity element of a O.P. subgroup G of Ω_1^F of the first kind there exists an inner automorphism of Ω^F which maps G on itself and carries g_1 to g_2.*

PROOF. (I) In two conjugate O.P. subgroups of Ω_1^F of the first kind every element different from the identity element of the first subgroup is conjugate with respect to Ω^F to any element different from the identity of the second subgroup (being of the same type and having the common characteristic number 0). Using the relation (33) we can find a pair of such elements which satisfy the conditions of Theorem 2, and hence are conjugate with respect to Ω_1^F. By Lemma 1 the two subgroups are then conjugate with respect to Ω_1^F.

(II) The statement of this clause follows from the fact that in a subgroup of the first kind any two elements different from the identity are of the same type and have the common characteristic number 0, and the use of Theorem 1 and Lemma 1.

5. Embedding theorems for two-parameter analytic subgroups of Ω^F

5.1. A subgroup of Ω^F is said to be an analytic two-parameter subgroup of Ω^F (a T.P. subgroup) if its elements can be written as a two parameter family of Ω^F

$$F(z,a^1,a^2) = \sum_{q=1}^{\infty} f_q(a^1,a^2)z^q \tag{34}$$

where $\vec{a} = (a^1,a^2)$ ranges in some domain of c^2, the two-dimensional complex space, such that:

I) $F(z,a^1,a^2)$ is analytically dependent on $\vec{a} = (a^1,a^2)$; that is, $f_q(a^1,a^2)$ are analytic functions of a^1 and a^2 for $(a^1,a^2) \in D$, when $q = 1,2,\cdots$.

II) There exists a vector function $\vec{\phi}(\vec{a},\vec{b})$, that is, two scalar functions of 4 variables: $\phi^i(a^1,a^2,b^1,b^2)$ $i = 1,2$, analytic in a^1,a^2,b^1,b^2 for (a^1,a^2), $(b^1,b^2) \in D$ such that

$$F[F(z,a^1,a^2),b^1,b^2] = F[z,\phi^1(a^1,a^2,b^1,b^2),\phi^2(a^1,a^2,b^1,b^2)] \tag{35}$$

holds for every (a^1,a^2), $(b^1,b^2) \in D$.

III) The subgroup is not a O.P. subgroup of Ω^F.

The importance of two parameter analytic subgroups of Ω^F was indicated in [15], where it was shown that the number of parameters in an analytic subgroup of Ω^F can always be reduced to two at most.

5.2. It was shown in [15] that the set of T.P. subgroups of Ω^F is split into a countable number of disjoint classes, each denoted by a positive integer, where a subgroup of class n is globally isomorphic to the group

$$H_n(z,a^1,a^2) = \frac{(1+a^1)z}{\sqrt[n]{1+a^2z^n}} \tag{36}$$

for $n = 1,2,\cdots$. More precisely, we cite the following representation theorem

REPRESENTATION THEOREM [15]. *Every T.P. subgroup of Ω^F of class n has a representation of the form*

$$F(z,a^1,a^2) = \phi^{-1}\left[\frac{(1+a^1)\cdot\phi(z)}{\sqrt[n]{1+a^2[\phi(z)]^n}}\right] \tag{37}$$

where $\phi(z)\in\Omega_1^F$.

Moreover,

$$g(z,a^{1*},a^{2*}) = \psi^{-1}\left[\frac{(1+a^{1*})\cdot\psi(z)}{\sqrt[n]{1+a^{2*}[\psi(z)]^n}}\right] \tag{38}$$

where $\psi(z)\in\Omega_1^F$, is another representation of the subgroup (37) *if and only if*

$$\psi(z) = \frac{\phi(z)}{\sqrt[n]{1+k[\phi(z)]^n}} \tag{39}$$

for some complex k.

5.3. The following theorems characterize those O.P. subgroups of Ω^F which are embeddable in T.P. subgroups of Ω^F. First we consider the case of non-simply connected subgroups of Ω^F.

THEOREM 6. *For every non-simply connected O.P. subgroup of Ω^F and every positive integer n there exists a unique T.P. subgroup of Ω^F of class n which contains the O.P. subgroup; if the O.P. subgroup is contained in Ω so is the T.P. subgroup.*

PROOF. Every non-simply connected O.P. subgroup of Ω^F can be represented in the form (31); the subgroup is thus contained, for every positive integer n, in the T.P. subgroup of class n

$$F(z,a^1,a^2) = \phi^{-1}\left[\frac{(1+a^1)\cdot\phi(z)}{\sqrt[n]{1+a^2[\phi(z)]^n}}\right] \tag{40}$$

(as we have $\hat{F}(z,a) = F(z,a,0)$). We also note, that if the O.P. subgroup is contained in Ω, then $\phi\in\Omega$ and hence the T.P. subgroup (40) is contained in Ω.

It remains to show that the T.P. subgroup given by (40) is the unique T.P. subgroup of Ω^F of class n which contains the O.P. subgroup represented by (31). Suppose that the T.P. subgroup of class n having the representation

$$g(z,a^1,a^2) = \psi^{-1}\left[\frac{(1+a^1)\cdot\psi(z)}{\sqrt[n]{1+a^2[\psi(z)]^n}}\right] \tag{41}$$

where $\psi(z) \in \Omega_1^F$, contains the O.P. subgroup given by (31); that is, we have

$$\phi^{-1}[(1+a)\cdot\phi(z)] = \psi^{-1}\left[\frac{\{1+\theta_1(a)\}\cdot\psi(z)}{\sqrt[n]{1+\theta_2(a)[\psi(z)]^n}}\right]. \tag{42}$$

Comparing the coefficients of z on both sides of (42) we get $\theta_1(a) = a$; putting in (42) $a = 0$ we get $\theta_2(0) = 0$.

Differentiating both sides of (42) with respect to a and putting $a = 0$, we get

$$(\phi^{-1})'[\phi(z)]\cdot\phi(z) = (\psi^{-1})'[\psi(z)]\left\{\psi(z) - \frac{\theta_2'(o)}{n}[\psi(z)]^{n+1}\right\} \tag{43}$$

(43) may be rewritten in the form

$$\frac{\phi(z)}{\phi'(z)} = \frac{\psi(z)}{\psi'(z)}\left\{1 - \frac{k}{n}[\psi(z)]^n\right\} \tag{44}$$

where $k = \theta_2'(o)$. Formal integration of (44) in $Q[\Sigma^F]$ yields

$$\phi(z) = \frac{\psi(z)}{\sqrt[n]{1 - \frac{k}{n}[\psi(z)]^n}}, \tag{45}$$

hence the representation theorem for T.P. subgroups of Ω^F implies that the subgroup represented by (41) coincides with the subgroup represented by (40), and the proof is complete.

5.4. We turn now to the simply connected O.P. subgroups of Ω^F, which are the O.P. subgroups of Ω_1^F.

THEOREM 7. *A O. P. subgroup of Ω_1^F of the second kind is non-embeddable in a T.P. subgroup of Ω^F.*

PROOF. We see from (37) that all the elements of a T.P. subgroup of Ω^F, of class n which belong to Ω_1^F are

$$F(z,0,a^2) = \phi^{-1}\left[\frac{\phi(z)}{\sqrt[n]{1+a^2[\phi(z)]^n}}\right] \tag{46}$$

which, being conjugate to the elements of the subgroup

$$\tilde{H}_n(z,a) = \frac{z}{\sqrt[n]{1+az^n}} \tag{47}$$

have the characteristic number 0; as a O.P. subgroup of Ω_1^F of the second kind contains no element which has the characteristic number 0 the theorem is proved.

THEOREM 8. *A O.P. subgroup of* Ω_1^F *of the first kind is embeddable in a unique T.P. subgroup of* Ω^F. *If the O.P. subgroup is of type n, the T.P. subgroup is of class n. If the O.P. subgroup is contained in* Ω, *so is the T.P. subgroup.*

PROOF. A O.P. subgroup of the first kind and of type n is conjugate, by Theorem 4, to the subgroup $\tilde{H}_n(z,a)$ given by (47), that is, we have the representation

$$\tilde{F}(z,a) = \phi^{-1}\left[\frac{\phi(z)}{\sqrt[n]{1 + a[\phi(z)]^n}}\right]. \tag{48}$$

Moreover, by Theorem 5(I) we may take in (48) $\phi(z) \in \Omega_1^F$. The subgroup having the representation (48) is contained in the T.P. subgroup

$$F(z,a^1,a^2) = \phi^{-1}\left[\frac{(1 + a^1)\phi(z)}{\sqrt[n]{1 + a^2[\phi(z)]^n}}\right], \tag{49}$$

as we have: $\hat{F}(z,a) = F(z,0,a)$. If the O.P. subgroup is contained in Ω we have $\phi(z) \in \Omega$ and hence the T.P. subgroup belongs to Ω.

It remains to show, that the T.P. subgroup given by (49) is the unique T.P. subgroup which contains the O.P. subgroup represented by (48). We note, that the O.P. subgroup (48), being of type n, cannot belong to any T.P. subgroup of class m with $m \neq n$ (as all the elements of Ω_1^F contained in a T.P. subgroup of class m are of type m).

Suppose that the O.P. subgroup (48) is contained in the T.P. subgroup of class n given by

$$g(z,a^1,a^2) = \psi^{-1}\left[\frac{(1 + a^1)\psi(z)}{\sqrt[n]{1 + a^2[\psi(z)]^n}}\right], \quad \psi(z) \in \Omega_1^F, \tag{50}$$

that is, we have

$$\phi^{-1}\left[\frac{\phi(z)}{\sqrt[n]{1 + a[\phi(z)]^n}}\right] = \psi^{-1}\left[\frac{\{1 + \theta_1(a)\}\psi(z)}{\sqrt[n]{1 + \theta_2(a)[\psi(z)]^n}}\right]. \tag{51}$$

Equating the coefficients of z on both sides of (51) we get $\theta_1(a) \equiv 0$. Putting $a = 0$ in (51) we get $\theta_2(0) = 0$. Differentiating both sides of (51) with respect to a and putting $a = 0$ we get

$$-\frac{1}{n}(\phi^{-1})'[\phi(z)] \cdot [\phi(z)]^{n+1} = (\psi^{-1})'[\psi(z)][\psi(z)]^{n+1} \cdot \left[-\frac{\theta_2'(o)}{n}\right].$$

The last equality may be rewritten in the form

(52)
$$-\frac{1}{n}\frac{[\phi(z)]^{n+1}}{\phi'(z)} = -\frac{\theta_2'(o)}{n}\frac{[\psi(z)]^{n+1}}{\psi'(z)}.$$

Comparing the coefficients of z^{n+1} on both sides of (52) we get $\theta_2'(o) = 1$. Formal integration of (52) in $Q[\Sigma^F]$ yields

(53)
$$\frac{1}{[\phi(z)]^n} = \frac{1}{[\psi(z)]^n} + k,$$

hence according to the representation theorem the subgroups represented by (49) and (50) coincide, and the theorem is proved.

5.5 Let G be a group, and $g \in G$. By $C(g)$ we denote the centralizator of g in G that is
$$C(g) = \{h \mid h \in G,\, hg = gh\}.$$

Baker [2] constructed the centralizators in Ω^F of the elements of Ω_1^F. Baker's result can be stated as follows

Let $F(z) \in \Omega_1^F$ be of type n, and ω a primitive root of unity of order n.

There exists a unique power series $H(z) = \sum_{q=1}^{\infty} h_q z^q$ such that

I) $H^{[n]}(z) = z$ II) $H(z) \in C(F)$ III) $h_1 = \omega$.

Moreover, $C(F)$ consists of the series of the form

$$\tilde{F}(z,s),\quad H[\tilde{F}(z,s)],\quad H^{[2]}[\tilde{F}(z,s)],\cdots,H^{[n-1]}[\tilde{F}(z,s)]$$

where $\tilde{F}(z,s)$ is the unique O.P. subgroup of Ω^F which contains $F(z)$.

$C(F)$ is a commutative group for every $F(z) \in \Omega_1^F$, $F(z) \neq z$.

We shall need the following lemma

LEMMA 2. *A T.P. subgroup of Ω^F contains with every element $F(z) \in \Omega_1^F$, $F(z) \neq z$ the centralizator $C(F)$.*

PROOF. Let $F(z) \in \Omega_1^F$, $F(z) \neq z$ belong to the T.P. subgroup of Ω^F

(54)
$$F(z,a^1,a^2) = \phi^{-1}\left[\frac{(1+a^1)\cdot\phi(z)}{\sqrt[n]{1+a^2[\phi(z)]^n}}\right]$$

that is, we have

(55)
$$F(z) = \phi^{-1}\left[\frac{\phi(z)}{\sqrt[n]{1+k[\phi(z)]^n}}\right]$$

for some complex k. Let ω be a primitive root of unity of order n. Consider the series: $H(z) = \phi^{-1}[\omega\phi(z)]$. We clearly have $H^{[n]}(z) = z$. We have also

$H(z) = F(z, \omega - 1, 0)$ that is, $H(z)$ belongs to the T.P. subgroup (54). By direct calculation we find that $H(z) \in C(F)$; as the unique O.P. subgroup which contains $F(z)$ is contained in the T.P. subgroup, $C(F)$ is also contained in the T.P. subgroup, by Baker's result.

5.6. We consider now the problem of embedding a given series of Ω^F in a T.P. subgroup of Ω^F.

THEOREM 9. (I) *A series $F(z) = \sum_{q=1}^{\infty} f_q z^q \in \Omega^F$ with $f_1^m \neq 1$ for all positive integers m can be embedded, for every positive integer n, in a unique T.P. subgroup of Ω^F of class n. The T.P. subgroup is contained in Ω iff $F(z) \in \Omega^A$.*

(II) *A series $F(z) = \sum_{q=1}^{\infty} f_q z^q \in \Omega^F$ such that for some positive integer m $F^{[m]}(z) = z$, with $f_1 \neq 1$, can be embedded in infinitely many T.P. subgroups of class n, for every positive integer n. If $F(z) \in \Omega$, then infinitely many of the T.P. subgroups are contained in Ω.*

(III) *A series $F(z) \in \Omega_1^F$, $F(z) \neq z$, can be embedded in a T.P. subgroup of Ω^F iff it has the characteristic number 0; $F(z) \in \Omega_1^F$ with $\rho(F) = 0$ can be embedded in a unique T.P. subgroup of Ω^F, which is of class n iff $\tau(F) = n$, and is contained in Ω iff $F(z) \in \Omega_1^A$.*

(IV) *A series $F(z) = \sum_{q=1}^{\infty} f_q z^q \in \Omega^F$ with f_1 a primitive root unity of order $n > 1$, but $F^{[n]}(z) \neq z$, is embeddable in a T.P. subgroup of Ω^F iff $F^{[n]}(z)$ (which belongs to Ω_1^F) has the characteristic number 0. When the series $F(z)$ is embeddable in a T.P. subgroup, the T.P. subgroup is unique and is of class m, $n \mid m$; it is a subgroup of Ω iff $F^{[n]}(z) \in \Omega_1^A$.*

NOTE. The series of the form discussed in IV are non-embeddable in a O.P. subgroup of Ω^F([5], [13]).

PROOF. (I) A series of the form under consideration can be embedded in a unique O.P. subgroup of Ω^F [9], which belongs to Ω iff $F(z) \in \Omega^A$; by Theorem 6 this subgroup can be embedded in a unique T.P. subgroup of class n, for every positive integer n.

In order to complete the proof we have to show that a T.P. subgroup of Ω^F contains with every element $F(z) = \sum_{q=1}^{\infty} f_q z^q$ where $f_1^m \neq 1$ for all positive integers m, the O.P. subgroup of Ω^F which contains it.

Suppose that $F(z)$ is contained in the T.P. subgroup

$$F(z, a^1, a^2) = \phi^{-1}\left[\frac{(1 + a^1)\phi(z)}{\sqrt[n]{1 + a^2[\phi(z)]^n}}\right].$$

That is, for some a_0^1, a_0^2 we have $F(z) = F(z, a_0^1, a_0^2)$. Consider the O.P. subgroup of Ω^F defined by $\hat{F}(z,a) = F[z,a,k\{(1+a)^n - 1\}]$ where k is defined by $k\{(1+a_0^1)^n - 1\} = a_0^2$ (as $1 + a_0^1 = f_1$, which is not a root of unity k is defined); we have $\hat{F}(z,a_0^1) = F(z,a_0^1,a_0^2) = F(z)$, and $\hat{F}(z,a)$ is contained in the T.P. subgroup.

(II) A series of this form can be embedded in a O.P. subgroup of Ω^F, which belongs to Ω if $F(z) \in \Omega$ ([5], [13]). We note that if $F(z)$ is contained in the O.P. subgroup $\hat{F}(z, a) = \phi^{-1}[(1+a)\phi(z)]$ it is also contained in any O.P. subgroup of the form: $\hat{G}(z,a) = \psi^{-1}[(1+a)\psi(z)]$, where $\psi(z) = \Gamma[\phi(z)]$,

$$\Gamma(z) = \sum_{k=0}^{\infty} \gamma_{km+1} z^{km+1}.$$

According to Theorem 6 each of these O.P. subgroups can be embedded in a T.P. subgroup of class n, for each n.

(III) As the intersection of a T.P. subgroup of Ω^F with Ω_1^F is a O.P. subgroup, the series $F(z)$ is embeddable in a T.P. subgroup of Ω^F iff the unique O.P. subgroup which contains it is embeddable in a T.P. subgroup. The statement of this clause follows now from Theorems 3, 7 and 8.

(IV) If $F(z)$ belongs to a T.P. subgroup, so does $F^{[n]}(z)$; as $F(z) \in C(F^{[n]})$ the converse is also true, by Lemma 2. The statement follows now from III and from the fact, that if $F(z)$ has the form indicated in the statement of the theorem, $F^{[n]}(z)$ is of type m, $n \mid m$, which follows from Baker's result, cited above in 5.5.

6. Normalizators and algebraic characterization of two-parameter analytic subgroups of Ω^F

6.1. Let G be a group, H a subgroup of G. By the normalizator of H in G we call the maximal subgroup of G in which H is normal, that is

$$N(H) = \{g \in G \mid \forall h \in H \ \ g^{-1}hg \in H\}.$$

The centralizator of H in G is denoted by

$$C(H) = \{g \in G \mid \forall h \in H \ \ gh = hg\}.$$

THEOREM 10. (I) *For any non-simply connected O.P. subgroup H of Ω^F, $N(H) = H$.*

(II) *For any O.P. subgroup H of Ω_1^F of the first kind $N(H) = T(H)$, where $T(H)$ is the unique T.P. subgroup of Ω^F which contains H.*

(III) *For any O.P. subgroup H of Ω_1^F of the second kind we have $N(H) = C(H)$.*

(IV) *For any T.P. subgroup H of Ω^F we have $N(H) = H$.*

PROOF. (I) Let the non-simply connected O.P. subgroup H of Ω^F be represented by: $\hat{F}(z,a) = \phi^{-1}[(1+a)\cdot\phi(z)]$, and let $g(z)\in N(H)$. Let a_0 be such that $1+a_0$ is not a root of unity. Then there exists a complex number a_1 such that $\hat{F}[g(z),a_0] = g[\hat{F}(z,a_1)]$. Comparing the coefficients of z in the last equality we get $a_1 = a_0$. Using the representation of $\hat{F}(z,a)$ the last equality can be rewritten in the form

$$\phi^{-1}\{(1+a_0)\cdot\phi[g(z)]\} = g\{\phi^{-1}[(1+a_0)\cdot\phi(z)]\}.$$

Denoting $1+a_0 = k$, $\Gamma(z) = \phi\{g[\phi^{-1}(z)]\}$, we get $k\Gamma(z) = \Gamma(kz)$ which implies, as k is not a root of unity, $\Gamma(z) = \alpha z$ for some complex α, and hence $g(z) = \phi^{-1}[\alpha\phi(z)]$, that is $g(z)\in H$. We have thus shown $N(H)\subset H$; as we always have $H\subset N(H)$ we get $N(H) = H$.

(II) Let H be a O.P. subgroup of Ω_1^F of the first kind, and let $T(H)$, the unique T.P. subgroup which contains H, have the representation (37). H admits then the representation

$$\hat{F}(z,s) = \phi^{-1}\left[\frac{\phi(z)}{\sqrt[n]{1+s[\phi(z)]^n}}\right] \tag{56}$$

As H is a normal subgroup of $T(H)$, we have $T(H)\subset N(H)$. To show the converse, suppose that $g(z)\in N(H)$; that is there exists a function $\theta(s)$ such that we have $g[\tilde{F}(z,s)] = \tilde{F}[g(z),\theta(s)]$, that is

$$g\left\{\phi^{-1}\left[\frac{\phi(z)}{\sqrt[n]{1+s[\phi(z)]^n}}\right]\right\} = \phi^{-1}\left[\frac{\phi\{g(z)\}}{\sqrt[n]{1+\theta(s)[\phi\{g(z)\}]^n}}\right]. \tag{57}$$

Putting in (57) $s = 0$ we find: $\theta(0) = 0$. Denoting $\Gamma(z) = \phi[g(z)]$ we can rewrite (57) in the form

$$\phi^{-1}\left[\frac{\phi(z)}{\sqrt[n]{1+s[\phi(z)]^n}}\right] = \Gamma^{-1}\left[\frac{\Gamma(z)}{\sqrt[n]{1+\theta(s)[\Gamma(z)]^n}}\right]. \tag{58}$$

Differentiating both sides of (58) with respect to s and putting $s = 0$ we get:

$$-\frac{1}{n}(\phi^{-1})'[\phi(z)]\cdot[\phi(z)]^{n+1} = -\frac{\theta'(o)}{n}(\Gamma^{-1})'[\Gamma(z)]\cdot[\Gamma(z)]^{n+1}.$$

The last equation can be rewritten in the form:

$$\phi'(z)[\phi(z)]^{-n-1} = [\theta'(o)]^{-1}\Gamma'(z)[\Gamma(z)]^{-n-1}.$$

Formal integration of the last equation and the definition of $\Gamma(z)$ yield

$$g(z) = \phi^{-1}\left[\frac{\alpha\phi(z)}{\sqrt[n]{1 + c[\phi(z)]^n}}\right]$$

and hence $g(z) \in T(H)$.

(III) It is clear that $C(H) \subset N(H)$. To show the inclusion in the opposite direction, let $g \in N(H)$. For every $f \in H$ we have $g^{-1} \circ f \circ g \in H$; but as in a simply connected O.P. subgroup of Ω^F of the second kind each element has a different characteristic number, f can be conjugate only to itself, that is, $g^{-1} \circ f \circ g = f$, hence $g \in C(H)$.

(IV) Let H be a T.P. subgroup of Ω^F. We always have $H \subset N(H)$. Let H have the representation (37). We note that $H \cap \Omega_1^F = G$ is a O.P. subgroup of Ω_1^F given by (56), and we have $H = T(G)$. As Ω_1^F is self-conjugate, we have for every $g \in N(H)$, $g \in N(G)$. Hence by part II $g \in T(G) = H$, so that $N(H) \subset H$ and the proof is complete.

6.2. Hadamard [8], Baker [2] and the author [13] treated the problem of giving an algebraic characterization to the class of the one-parameter analytic subgroups of Ω^F, without reference to analytic or continuous dependence on a parameter. The class of the non-simply connected O.P. subgroups of Ω^F coincides with the class of those maximal commutative subgroups of Ω^F, which contain n distinct roots of unity of order n for every positive integer n. The class of the simply connected O.P. subgroups of Ω^F coincides with the class of maximal commutative subgroups of Ω_1^F.

Theorem 10 gives us the corresponding characterization for two parameter groups, the proof of which is obvious.

THEOREM 11. *The class of T.P. subgroups of Ω^F coincides with the class of noncommutative normalizators of O.P. sugbroups of Ω^F; or, equivalently, with the class of normalizators of O.P. subgroups of Ω_1^F of the first kind.*

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